

**Commutation Relation**

$$\begin{aligned}
 [\hat{A}, \hat{B}] &= -[\hat{B}, \hat{A}] & [\hat{A}\hat{B}, \hat{C}] &= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} \\
 [\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] &\equiv 0 \\
 e^{\hat{B}}\hat{A}e^{-\hat{B}} &= \hat{A} + [\hat{B}, \hat{A}] + \dots + \frac{1}{n!}[\hat{B}, [\hat{B}, \dots, [\hat{B}, \hat{A}]\dots]] + \dots & e^{\hat{B}}F(\hat{A})e^{-\hat{B}} &= F(e^{\hat{B}}\hat{A}e^{-\hat{B}}) \\
 [\hat{A}, \hat{B}] = \text{C-num} &\implies [F(\hat{A}), \hat{B}] = F'(\hat{A})[\hat{A}, \hat{B}] & e^{\hat{A}+\hat{B}} &= e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \\
 \Delta A \Delta B &\geq \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle | & \text{where } \Delta A &\equiv \sqrt{\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \\
 [\hat{p}, \hat{x}] &= -i\hbar
 \end{aligned}$$

$h = 2\pi\hbar$  : phase space volume for one quantum state in 1-dim.

Minimum Wave Packet:  $\Delta A \Delta B = \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle |$

$$\begin{aligned}
 \langle [\hat{A}, \hat{B}] \rangle (\hat{A} - \langle \hat{A} \rangle) - 2(\Delta A)^2 (\hat{B} - \langle \hat{B} \rangle) | \psi \rangle &= 0 \\
 \hat{A} = \hat{x}, \hat{B} = \hat{p} &\implies \psi = \text{Const} \times \exp\left[\frac{i}{\hbar} \langle \hat{p} \rangle (\hat{x} - \langle \hat{x} \rangle)\right] \exp\left[-\frac{(\hat{x} - \langle \hat{x} \rangle)^2}{(2\Delta x)^2}\right]
 \end{aligned}$$

**Ket**  $|\psi\rangle$

$$\psi(x, t) = \langle x | \psi \rangle \quad \langle x | x' \rangle = \delta(x - x') \quad \sum_x |x\rangle \langle x| = 1$$

**Expansion with Eigen Ket**  $\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle$

$$\begin{aligned}
 |\psi, t\rangle &= \sum_n |\phi_n\rangle \langle \phi_n | e^{-\frac{i}{\hbar} E_n (t-t')} | \psi, t' \rangle \\
 &= \sum_{n, x, x'} |x\rangle \phi_n(x) \phi_n^*(x') e^{-\frac{i}{\hbar} E_n (t-t')} \langle x' | \psi, t' \rangle \\
 &= \sum_{n, x, x'} |x\rangle G(x, t; x', t') \langle x' | \psi, t' \rangle
 \end{aligned}$$

$$\langle x | \hat{A} | \psi \rangle = \sum_{x'} \langle x | \hat{A} | x' \rangle \langle x' | \psi \rangle = \sum_{x'} A_{xx'} \psi(x')$$

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \quad \text{Expected value}$$

**Schrödinger Equation**

$$\begin{aligned}
 \hat{p} &= -i\hbar \nabla & \hat{x} &= \mathbf{x} & \text{Coordinate representation} \\
 \hat{H} | \psi, t \rangle &= i\hbar \frac{\partial}{\partial t} | \psi, t \rangle & \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right] \psi(\mathbf{x}, t) &= i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) \\
 \psi(\mathbf{x}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar} (\mathbf{p} \cdot \mathbf{x} - Et)} & \text{Plane wave} \\
 \hat{p} &\rightarrow \hat{p} - q\mathbf{A} = -i\hbar \nabla - q\mathbf{A} & \text{with vector potential } \mathbf{B} = \nabla \times \mathbf{A}
 \end{aligned}$$

**Heisenberg Representation**

$$\hat{U}(t, t') = e^{-\frac{i}{\hbar} \hat{H}(t-t')} \quad \text{Evolution operator}$$

Time Evolution of Eigen State  $\hat{U}$  : Unitary  $\hat{U}^\dagger = \hat{U}^{-1}$

$$\hat{U} | \phi_n \rangle = e^{-\frac{i}{\hbar} E_n t} | \phi_n \rangle$$

$\hat{O}_S$  : Schrödinger representation  $\hat{O}_H$  : Heisenberg representation

$|\psi\rangle_{\text{H}} = |\psi, t_0\rangle_{\text{H}} \equiv |\psi(t=t_0)\rangle_{\text{S}} = \hat{U}^\dagger(t, t_0)|\psi(t)\rangle_{\text{S}}$  : State

$$\hat{O}_{\text{H}}(t) = \hat{U}^{-1}(t)\hat{O}_{\text{S}}\hat{U}(t) \quad i\hbar\frac{\partial}{\partial t}\hat{O}_{\text{H}}(t) = [\hat{O}_{\text{H}}(t), \hat{H}]$$

$$[\hat{A}_{\text{S}}, \hat{B}_{\text{S}}] = \hat{C}_{\text{S}} \quad \implies \quad [\hat{A}_{\text{H}}, \hat{B}_{\text{H}}] = \hat{C}_{\text{H}}$$

$$\hat{H}_{\text{H}} = \frac{\hat{\mathbf{p}}_{\text{H}}^2}{2m} + V(\hat{\mathbf{r}}_{\text{H}}) \quad \implies \quad \frac{d}{dt}\hat{\mathbf{r}}_{\text{H}} = \frac{\hat{\mathbf{p}}_{\text{H}}}{m} \quad \frac{d}{dt}\hat{\mathbf{p}}_{\text{H}} = -\frac{\partial}{\partial\hat{\mathbf{r}}_{\text{H}}}V(\hat{\mathbf{r}}_{\text{H}})$$

Wave Function in Heisenberg Representation

$\hat{x}_{\text{H}}(t)|x, t\rangle_{\text{H}} = x|x, t\rangle_{\text{H}}$ ,  $\hat{x}|x\rangle_{\text{S}} = x|x\rangle_{\text{S}}$  : Coordinate eigen ket at time  $t$

${}_{\text{H}}\langle x, t|\psi\rangle_{\text{H}} = {}_{\text{S}}\langle x|\psi, t\rangle_{\text{S}}$

**Interaction Representation**

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$\hat{O}_{\text{I}} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{O}_{\text{S}} e^{-\frac{i}{\hbar}\hat{H}_0 t} \quad |\psi(t)\rangle_{\text{I}} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi(t)\rangle_{\text{S}}$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_{\text{I}} \equiv \hat{V}_{\text{I}}|\psi(t)\rangle_{\text{I}} \quad \hat{V}_{\text{I}} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{V}_{\text{S}} e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$i\hbar\frac{\partial}{\partial t}\hat{O}_{\text{I}}(t) = [\hat{O}_{\text{I}}(t), \hat{H}_0]$$

$$[\hat{A}_{\text{S}}, \hat{B}_{\text{S}}] = \hat{C}_{\text{S}} \quad \implies \quad [\hat{A}_{\text{I}}, \hat{B}_{\text{I}}] = \hat{C}_{\text{I}}$$

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}_{\text{I}}^2}{2m} + V_0(\hat{\mathbf{r}}_{\text{I}}) \quad \implies \quad \frac{d}{dt}\hat{\mathbf{r}}_{\text{I}} = \frac{\hat{\mathbf{p}}_{\text{I}}}{m} \quad \frac{d}{dt}\hat{\mathbf{p}}_{\text{I}} = -\frac{\partial}{\partial\hat{\mathbf{r}}_{\text{I}}}V_0(\hat{\mathbf{r}}_{\text{I}})$$

**Evolution Operator**

$$\hat{U}(t'', t') = \exp\left\{-\frac{i}{\hbar}\hat{H}(t'' - t')\right\}$$

$\Psi'$ : A state in Schrödinger picture at  $t'$ .

$\Psi''$ : The state where  $\Psi'$  will turn out to be at  $t''$ .

$$\Psi'' = U(t'', t')\Psi'$$

Matrix elements:  $\langle q''|U(t'', t')|q'\rangle = \langle q'', t''|q', t'\rangle$

**Probability Flux**

$$\mathbf{j}(\mathbf{x}, t) = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - (\nabla\psi^*)\psi) \quad \psi = \sqrt{\rho(\mathbf{x}, t)}e^{\frac{i}{\hbar}S(\mathbf{x}, t)} \Rightarrow \mathbf{j} = \frac{\rho}{m}\nabla S$$

$$= \frac{\hbar}{m}\text{Im}(\psi^*\nabla\psi)$$

$$\rho(\mathbf{x}, t) = \psi^*\psi \quad \frac{\partial\rho}{\partial t} + \nabla\mathbf{j} = 0 : \text{Equation of continuity}$$

**Harmonic Oscillator**

1-Dimensional Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$[\hat{a}^\dagger\hat{a}, \hat{a}] = -\hat{a} \quad [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

Eigen Function

$$\hat{a}^\dagger = -\frac{i}{\sqrt{2m\hbar\omega}} \exp\left(\frac{1}{2}\alpha^2 \hat{x}^2\right) \hat{p} \exp\left(-\frac{1}{2}\alpha^2 \hat{x}^2\right) \quad \alpha \equiv \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi_n(x) = \sqrt{\frac{\alpha}{n!2^n\sqrt{\pi}}} H_n(\alpha x) \exp\left(-\frac{1}{2}\alpha^2 \hat{x}^2\right)$$

$$H_n(z) \equiv (-1)^n \exp(z^2) \left(\frac{d}{dx}\right)^n \exp(-z^2) : \text{ Hermite polynomials}$$

### Coherent State

$$\hat{a}|a\rangle = a|a\rangle \quad a \in \mathbb{C}$$

$$|a\rangle = \exp(a\hat{a}^\dagger - a^*\hat{a})|0\rangle = \exp\left(-\frac{1}{2}|a|^2\right) \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |n\rangle \quad \langle a|a'\rangle = \exp\left(-\frac{1}{2}|a - a'|^2\right)$$

$$\int da |a\rangle \langle a| = \pi : \text{ Over complete}$$

$$a\hat{a}^\dagger - a^*\hat{a} = \frac{i}{\hbar}(p\hat{x} - x\hat{p}) \quad \text{where } a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i\frac{p}{m\omega}\right)$$

$$\langle a|\hat{x}|a\rangle = x \quad \langle a|\hat{p}|a\rangle = p$$

$$\Delta x^2 \equiv \langle a|(\hat{x} - \langle \hat{x} \rangle)^2|a\rangle = \langle 0|\hat{x}^2|0\rangle = \frac{1}{2\alpha^2}$$

$$\Delta p^2 \equiv \langle a|(\hat{p} - \langle \hat{p} \rangle)^2|a\rangle = \langle 0|\hat{p}^2|0\rangle = \frac{\hbar^2\alpha^2}{2} \quad \alpha \equiv \frac{m\omega}{\hbar}$$

### Time Evolution of Coherent State

$$|a(t)\rangle = \exp\left(-\frac{i}{2}\omega t\right) \exp[a(t)\hat{a}^\dagger - a^*(t)\hat{a}]|0\rangle \quad a(t) = a e^{-i\omega t}$$

$$\begin{aligned} x(t) \equiv \langle a(t)|\hat{x}|a(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}} (a^*(t) + a(t)) \\ &= \sqrt{\frac{2\hbar}{m\omega}} |a| \cos(\omega t + \delta) \end{aligned}$$

$$\begin{aligned} p(t) \equiv \langle a(t)|\hat{p}|a(t)\rangle &= i\sqrt{\frac{\hbar m\omega}{2}} (a^*(t) - a(t)) \\ &= -\sqrt{2\hbar m\omega} |a| \sin(\omega t + \delta) \end{aligned}$$

### Orbital Angular Momentum of Particle

$$\hbar \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \quad [\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk} \hat{L}_k$$

$$[\hat{L}_\pm, \hat{L}_z] = \mp \hat{L}_\pm \quad [\hat{L}^2, \hat{L}_z] = [\hat{L}^2, \hat{L}_\pm] = 0 \quad \hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$\hat{L}_- \hat{L}_+ = \hat{L}^2 - \hat{L}_z(\hat{L}_z + 1) \quad \hat{L}_+ \hat{L}_- = \hat{L}^2 - \hat{L}_z(\hat{L}_z - 1)$$

$$\hat{L}_x = -i \left( -\sin\varphi \frac{\partial}{\partial\theta} - \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} \right) \quad \hat{L}_y = -i \left( \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \right)$$

$$\hat{L}_+ = e^{i\varphi} \left( \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \quad \hat{L}_- = e^{-i\varphi} \left( -\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$\hat{L}_z = -i \frac{\partial}{\partial\varphi} \quad \hat{L}^2 = -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) - \frac{1}{\sin^2\theta} \left( \frac{\partial}{\partial\varphi} \right)^2$$

### Eigen State

$$\hat{L}^2|l, m\rangle = l(l+1)|l, m\rangle \quad \hat{L}_z|l, m\rangle = m|l, m\rangle$$

### Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U(|\mathbf{r}|) = -\frac{\hbar^2}{2m} \frac{1}{r} \left( \frac{\partial}{\partial r} \right)^2 r + \frac{\hbar^2 \hat{\mathbf{L}}^2}{2m r^2} + U(|\mathbf{r}|)$$

If we let  $\psi(r, \theta, \varphi) = \frac{u(r)}{r} Y_{lm}(\theta, \varphi)$

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{d}{dr} \right)^2 + \frac{\hbar^2 l(l+1)}{2m r^2} + U(|\mathbf{r}|) \right] u(r) = E u(r)$$

$$Y_{lm}(\theta, \varphi) \equiv \frac{1}{\sqrt{2\pi}} e^{im\varphi} (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}} P_l^{|m|}(\cos\theta)$$

$$\text{where } P_l^m(x) = \frac{(-1)^l}{2^l l!} (1-x^2)^{m/2} \left( \frac{d}{dx} \right)^{l+m} (1-x^2)^l$$

$$\hat{L}_{\pm}|l, m\rangle = \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle$$

### Spin Angular Momentum

Pauli Spin Matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k = \begin{pmatrix} 1 & i\sigma_3 & -i\sigma_2 \\ -i\sigma_3 & 1 & i\sigma_1 \\ i\sigma_2 & -i\sigma_1 & 1 \end{pmatrix} \quad \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}) \quad (\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2$$

$$\{\boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A})\} \{\boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A})\} \varphi = (-i\nabla - q\mathbf{A})^2 \varphi - q\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}) \varphi$$

$$\exp(i\theta\sigma_i) = \cos\theta + i\sigma_i \sin\theta$$

### Composition of Angular Momentum

Clebsch-Gordan Coefficient

$j_1, j_2$  : given

$$|j_1 j_2; m_1 m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad \text{or} \quad |j_1 j_2; j m\rangle$$

$$|j_1 j_2; j m\rangle = \sum_{m_1, m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle : \text{CG coefficient}$$

$$|j_1 j_2; j m\rangle = \left[ \frac{(j+m)!}{(2j)!(j-m)!} \right]^{1/2} \hat{J}_-^{j-m} |j_1 j_2; j j\rangle$$

Racah's formula

$$\begin{aligned} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle &= \sqrt{2j+1} \left[ \frac{(-j+j_1+j_2)! \cdot (j+j_1-j_2)! \cdot (j-j_1+j_2)!}{(j+j_1+j_2+1)!} \right]^{1/2} \\ &\times [(j_1+m_1)! \cdot (j_1-m_1)! \cdot (j_2+m_2)! \cdot (j_2-m_2)! \cdot (j_2+m_2)! \cdot (j_2-m_2)!]^{1/2} \\ &\times \sum_k \frac{(-1)^k}{k!} [(j_1+j_2-j-k)! \cdot (j_1-m_1-k)! \cdot (j_2+m_2-k)! \\ &\cdot (j-j_2+m_1+k)! \cdot (j-j_1-m_2+k)!]^{-1} \end{aligned}$$

### d-function

$$d_{n,m}^j(\theta) \equiv \langle j, n | e^{-i\theta \hat{J}_y} | j, m\rangle$$

$$d_{n,m}^j(\theta) = d_{-m,-n}^j(\theta) = (-1)^{n-m} d_{m,n}^j(\theta)$$

$$d_{n,m}^j(\theta) = (-1)^{j+n} d_{n,-m}^j(\pi - \theta)$$

$$\begin{pmatrix} d_{1/2,1/2}^{1/2} & d_{1/2,-1/2}^{1/2} \\ d_{-1/2,1/2}^{1/2} & d_{-1/2,-1/2}^{1/2} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

$$\begin{pmatrix} d_{1,1}^1 & d_{1,0}^1 & d_{1,-1}^1 \\ d_{0,1}^1 & d_{0,0}^1 & d_{0,-1}^1 \\ d_{-1,1}^1 & d_{-1,0}^1 & d_{-1,-1}^1 \end{pmatrix} = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & -\cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

**Perturbation Theory** ( $\hbar = 1$ )

$$H_0|n\rangle = i\frac{\partial}{\partial t}|n\rangle \quad |n, t\rangle = U_0(t, t_0)|n, t_0\rangle = e^{-iE_n(t-t_0)}|n, t_0\rangle$$

$$(H_0 + V)|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle \quad |\psi, t\rangle = U(t, t_0)|\psi, t_0\rangle$$

Integral equation on  $U(t, t_0)$

$$U(t, t_0) = U_0(t, t_0) - i \int_{t_0}^t dt' U_0(t, t') V(t') U(t', t_0)$$

**Perturbation Expansion**  $U(t, t_0) = U_0(t, t_0) + U^{(1)}(t, t_0) + U^{(2)}(t, t_0) + \dots$

$$U^{(1)}(t, t_0) = -i \int_{t_0}^t dt' U_0(t, t') V(t') U_0(t', t_0)$$

$$U^{(2)}(t, t_0) = \frac{(-i)^2}{2!} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' U_0(t, t') V(t') U_0(t', t'') V(t'') U_0(t'', t_0)$$

$$= \frac{(-i)^2}{2!} \int_{t_0}^t dt' \int_{t_0}^t dt'' T[U_0(t, t') V(t') U_0(t', t'') V(t'') U_0(t'', t_0)]$$

$$U(t, t_0) = U_0(t, t_0) T \exp \left[ -i \int_{t_0}^t dt' U_0(t_0, t') V(t') U_0(t', t_0) \right]$$

where  $T$  is time ordering operator (T-product)

**Transition Amplitude**

$$A_{fi}^{(1)} \equiv \langle f, t | U^{(1)}(t, t_0) | i, t_0 \rangle = -i \int_{t_0}^t dt' V_{fi}(t') e^{i(E_f - E_i)(t' - t_0)}$$

$$V_{fi}(t') \equiv \langle f, t_0 | V(t') | i, t_0 \rangle = \int dx' \phi_f^*(x', t_0) V(x', t') \phi_i(x', t_0)$$

$$A_{fi}^{(2)} = (-i)^2 \sum_n \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V_{fn}(t') e^{i(E_f - E_n)(t' - t_0)} V_{ni}(t'') e^{i(E_n - E_i)(t'' - t_0)}$$

In case that  $V$  is independent of  $t$

$$t_0 \rightarrow -\infty \quad t \rightarrow +\infty$$

$$T_{fi} \equiv \langle f, \infty | U(\infty, -\infty) | i, -\infty \rangle$$

$$T_{fi}^{(1)} = -2\pi i V_{fi} \delta(E_f - E_i)$$

$$T_{fi}^{(2)} = -2\pi i \sum_n V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni} \delta(E_f - E_i)$$

**Fermi Golden Rule**

$$W_{i \rightarrow f} = \frac{2\pi}{\hbar} |M_{fi}|^2 \rho \quad M_{fi} = V_{fi} + \sum_{n \neq i} \frac{V_{fn} V_{ni}}{E_i - E_n} + \dots \quad V_{fi} \equiv \langle f | \hat{V} | i \rangle$$

$$\rho: \text{Number of the final state per unit energy } \frac{dn}{dE} = \delta(E_f - E_i) \frac{V d^3 p}{(2\pi\hbar)^3}$$