Acceleration in Spherical Coordinates

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

Miscellaneous Formulas

$$\vec{F} = -\boldsymbol{\nabla} U \qquad \dot{\vec{p}} = \vec{F}$$

$\vec{l} = \vec{r} \times \vec{p}$ Angular momentum

$\vec{N} = \vec{r} \times \vec{F}$ Moment of force $\dot{\vec{l}} = \vec{N}$

$\mu = \dfrac{mM}{m+M}$ Reduced mass

Rotating Coordinate System

Consider S' that rotates about an axis in S by angular velocity $\vec{\omega}$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\omega} \times \vec{A} \qquad \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\frac{d^2\vec{A}}{dt^2} = \frac{d'^2\vec{A}}{dt^2} + 2\,\vec{\omega} \times \frac{d'\vec{A}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{A}) + \dot{\vec{\omega}} \times \vec{A} \qquad \dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} = \frac{d'\vec{\omega}}{dt}$$

Moment of Inertia

$I = \displaystyle\int \rho r^2 dv$ Moment of inertia about Line ℓ r: Distance from Line ℓ

Inertia Tensor

$$I_{xx} = \int \rho(y^2+z^2)dv \qquad I_{yy} = \int \rho(z^2+x^2)dv \qquad I_{zz} = \int \rho(x^2+y^2)dv$$

$$I_{xy} = -\int \rho xydv \qquad I_{yz} = -\int \rho yzdv \qquad I_{zx} = -\int \rho zxdv$$

$$L_i = I_{ij}\omega_j \qquad I_{ij} = I_{ji}$$

Moment of inertia about a line through origin O with direction cosine (λ, μ, ν).

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} \qquad \lambda^2 + \mu^2 + \nu^2 = 1$$

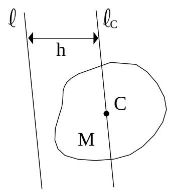
$$\begin{aligned} I &= I_{xx}\,\lambda^2 + I_{yy}\,\mu^2 + I_{zz}\,\nu^2 + 2\,I_{xy}\,\lambda\,\mu + 2\,I_{yz}\,\mu\,\nu + 2\,I_{zx}\,\nu\,\lambda \\ &= I_{ij}\,\lambda_i\,\lambda_j \end{aligned}$$

where $\lambda_i = \lambda, \mu, \nu$

Moment of inertia about a parallel axis

ℓ_{C} : A straight line through center of gravity C parallel to line ℓ .
I : Moment of inertia about line ℓ
I_C : Moment of inertia about line ℓ_{C}

$$I = I_C + Mh^2$$



Principal Moment of Inertia

$$\vec{\rho} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Coordinate transformation $\vec{\rho} = U^{-1}\vec{r}$ $\qquad U^T = U^{-1}$

$$\vec{\lambda} = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}$$

Direction cosine vector $\vec{\lambda}' = U^{-1}\vec{\lambda}$

$$I = \vec{\lambda}^T \{I_{ij}\} \vec{\lambda} = \vec{\lambda}'^T U^T \{I_{ij}\} U \vec{\lambda}' = \vec{\lambda}'^T \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \vec{\lambda}'$$

Examples of principal moment of inertia about simple shapes

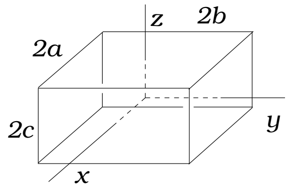
$$A = I_{xx} \qquad B = I_{yy} \qquad C = I_{zz}$$

Spherical shell $\quad A = B = C = \frac{2}{3} M a^2 \quad a$: Radius

Sphere $\quad A = B = C = \frac{2}{5} M a^2$

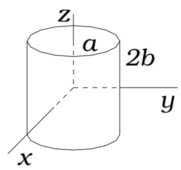
Rectangular pole

$$A = \frac{1}{3} M (b^2 + c^2)$$
$$B = \frac{1}{3} M (c^2 + a^2)$$
$$C = \frac{1}{3} M (a^2 + b^2)$$



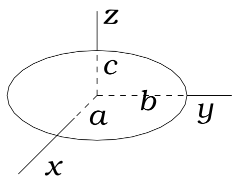
Circular cylinder

$$A = B = M\left(\frac{a^2}{4} + \frac{b^2}{3}\right)$$
$$C = \frac{1}{2} M a^2$$



Ellipsoidal

$$A = \frac{1}{5} M (b^2 + c^2)$$
$$B = \frac{1}{5} M (c^2 + a^2)$$
$$C = \frac{1}{5} M (a^2 + b^2)$$



Kinetic Energy of Rigid Body

$$K = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} I \omega^2 \qquad I : \text{Moment of inertia about the direction of } \vec{\omega}$$

Motion of Rigid Body around Fixed Point

Euler Equations

$$A \dot{\omega}_x - (B - C) \omega_y \omega_z = N_x \qquad A = I_{xx}$$
$$B \dot{\omega}_y - (C - A) \omega_z \omega_x = N_y \qquad B = I_{yy} \qquad \text{Principal moment of inertia}$$
$$C \dot{\omega}_z - (A - B) \omega_x \omega_y = N_z \qquad C = I_{zz}$$

$$\frac{dK}{dt} = \vec{\omega} \cdot \vec{N} : \text{ Enegry equation} \qquad K : \text{Kinetic energy}$$

$$\vec{L} \cdot \frac{d'\vec{L}}{dt} = \frac{1}{2} \frac{d'\vec{L}^2}{dt} = \frac{1}{2} \frac{d\vec{L}^2}{dt} = \vec{N} \cdot \vec{L}$$

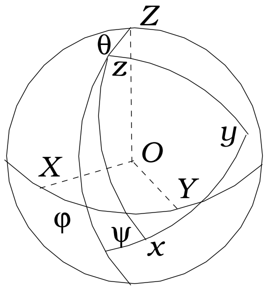
Euler Angles

OXYZ : Space fixed frame $\Rightarrow \vec{R}$

Oxyz : Body fixed frame $\Rightarrow \vec{r}$

$$\vec{r} = \begin{pmatrix} \cos\varphi\cos\theta\cos\psi - \sin\varphi\sin\psi & \sin\varphi\cos\theta\cos\psi + \cos\varphi\sin\psi & -\sin\theta\cos\psi \\ -\cos\varphi\cos\theta\sin\psi - \sin\varphi\cos\psi & -\sin\varphi\cos\theta\sin\psi + \cos\varphi\cos\psi & \sin\theta\sin\psi \\ \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \end{pmatrix}\vec{R}$$

$$\begin{aligned} \omega_x &= \dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi \\ \omega_y &= \dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi \\ \omega_z &= \dot\varphi\cos\theta + \dot\psi \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot\theta &= \omega_x\sin\psi + \omega_y\cos\psi \\ \dot\varphi &= \mathrm{cosec}\theta(-\omega_x\cos\psi + \omega_y\sin\psi) \\ \dot\psi &= \omega_z - \cot\theta(-\omega_x\cos\psi + \omega_y\sin\psi) \end{aligned}$$



$$\vec{R} \to \vec{r} : R_z(-\psi) \circ R_y(-\theta) \circ R_z(-\varphi)$$

Analytical Machanics of System of Particles

$L = T - U$: Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_r}\right) = \frac{\partial L}{\partial q_r} \qquad (r = 1, \cdots, f)$$

$L = T - q\varphi + q\vec{v}\cdot\vec{A}$: Lagrangian of charged particle in electric and magnetic field

Hamiltonian

$$p_r = \frac{\partial L}{\partial \dot q_r}$$

$$H(p_r, q_r) = \sum_{r=1}^{f} p_r \dot q_r - L(q_1, \cdots, q_f, \dot q_1, \cdots, \dot q_f, t)$$

$\dot q_r = \frac{\partial H}{\partial p_r} \qquad \dot p_r = -\frac{\partial H}{\partial q_r}$: Canonical Equations

$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi$: Hamiltonial of charged particle in EM field

$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\varphi^2\right) + V(r)$: Hamiltonian in spherical coordinate

Canonical Transformation

$$H(p, q, t) \to \bar{H}(P, Q, t)$$

Consider generating functions like $W = W(q_1, \cdots, q_f, Q_1, \cdots, Q_f, t) \to W(q, Q, t)$

Generating functions (+ sign for variables q and P, $-$ sign for variables Q and p)

$W(q, Q)$	$p = \frac{\partial W}{\partial q}$	$P = -\frac{\partial W}{\partial Q}$
$W(q, P)$	$p = \frac{\partial W}{\partial q}$	$Q = \frac{\partial W}{\partial P}$
$W(Q, p)$	$P = -\frac{\partial W}{\partial Q}$	$q = -\frac{\partial W}{\partial p}$
$W(p, P)$	$Q = \frac{\partial W}{\partial P}$	$q = -\frac{\partial W}{\partial p}$

$$\bar{H} = H + \frac{\partial W}{\partial t}$$

Hamilton-Jacobi Partial Differential Equation

i. Make Hamiltonian. $H(q,p,t)$

ii. Solve H-J Equation: $\frac{\partial}{\partial t}W + H\left(q, \frac{\partial W}{\partial q}, t\right) = 0$

Complete solution of H-J eq. will be $W(q,\alpha,t)$ where $\alpha = (\alpha_1, \cdots, \alpha_f)$ is constant.

iii. $p_r = \frac{\partial W}{\partial q_r}$ $\beta_r = \frac{\partial W}{\partial \alpha_r}$ α, β : Constants

Case that H doesn't contain t explicitly

i. Solve H-J equation: $H\left(q, \frac{\partial S}{\partial q}\right) = E$

In this case, separation of variables is usually applicable and E and S are

$$E = E(\alpha_1, \cdots, \alpha_f) \qquad S = S(q_1, \cdots, q_f, \alpha_1, \cdots, \alpha_f)$$

$E = \alpha_1 + \cdots + \alpha_f$ usually, or $E = \alpha_1$ in case to obtain trajectory.

ii. $p_r = \frac{\partial S}{\partial q_r}$ $\beta_r = -\frac{\partial E}{\partial \alpha_r}t + \frac{\partial S}{\partial \alpha_r}$

Poisson Bracket

$$\{u,v\} = \sum_r \left(\frac{\partial u}{\partial q_r}\frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r}\frac{\partial v}{\partial q_r}\right)$$

$$\{u,v\} = -\{v,u\} \qquad \{u,u\} = 0 \qquad \{u,vw\} = v\{u,w\} + \{u,v\}w$$

$$\{u_1+u_2, v_1+v_2\} = \{u_1,v_1\} + \{u_2,v_1\} + \{u_1,v_2\} + \{u_2,v_2\}$$

$$\{u,v,w\} \equiv \{u,\{v,w\}\} + \{v,\{w,u\}\} + \{w,\{u,v\}\} = 0 \qquad \text{(Jacobi identity)}$$

$$\dot{F} = \{F,H\} + \frac{\partial F}{\partial t} \qquad \dot{q} = \{q,H\} \qquad \dot{p} = \{p,H\}$$

Relativistic Mechanics

$$x^\mu = (ct, \vec{x}) \qquad u^\mu = \frac{dx^\mu}{d\tau} = (c\gamma, \gamma\vec{v}) \qquad \vec{v} = \frac{d\vec{x}}{dt} \qquad \frac{dt}{d\tau} = \gamma \qquad u^\mu u_\mu = c^2$$

$$p^\mu = m_0 u^\mu = \left(\frac{E}{c}, \vec{p}\right) \qquad m = m_0\gamma \qquad E = mc^2 \qquad \vec{p} = m\vec{v} \qquad p^\mu p_\mu = m_0^2$$

$$F^\mu = \frac{dp^\mu}{d\tau} = \left(\gamma\frac{\vec{F}\cdot\vec{v}}{c}, \gamma\vec{F}\right) \qquad \vec{F} = \frac{d\vec{p}}{dt} \qquad F^\mu u_\mu = 0$$

$$F^\mu = m_0 a^\mu \qquad a^\mu = \frac{du^\mu}{d\tau} = \gamma^4\left(\frac{\vec{v}\cdot\vec{a}}{c}, \vec{a} + \frac{\vec{v}\times(\vec{v}\times\vec{a})}{c^2}\right) \qquad \vec{a} \equiv \frac{d\vec{v}}{dt}$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \boldsymbol{\nabla}\right) = \left(\frac{\partial}{c\partial t}, \boldsymbol{\nabla}\right)$$

$$k^\mu = \left(\frac{\omega}{c}, \vec{k}\right) \qquad j^\mu = (c\rho, \vec{j}\,) \qquad A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$$

$$L = m_0c^2\left(1 - \sqrt{1-\beta^2}\right) - U \qquad H = mc^2 - m_0c^2 + U$$

Charged Particle

$$L = m_0c^2\left(1 - \sqrt{1-\beta^2}\right) - q\varphi + q\vec{v}\cdot\vec{A} \qquad H = \sqrt{m_0^2c^4 + c^2(\vec{p} - q\vec{A}\,)^2} - m_0c^2 + q\varphi$$

$$pc(\text{GeV}) = 0.3\cdot\rho(\text{m})\cdot B(\text{Tesla})$$